§6.5 Unitary and orthogonal operators

7. Prove that if T is a unitary operator on a finite dimensional inner product space V then T has a unitary square root.

Proof. Corollary 2 to Thm 6.18 $\Rightarrow$ $\exists$ an orthonormal basis β s.t.

$$[T]_\beta = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix} \qquad |\lambda_i| = 1.$$

$\Rightarrow \exists \mu_i$, s.t. $\mu_i^2 = \lambda_i$ $\quad |\mu_i| = 1$.

$$D := \begin{pmatrix} \mu_1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \mu_n \end{pmatrix} \quad \text{— unitary operator}$$

$U :=$ the matrix ~~s.t.~~ whose matrix rep w.r.t β is D.

$\Rightarrow U$ is unitary and $U^2 = T$

10. A $n \times n$ real symmetric matrix or complex normal matrix. Prove that $\operatorname{tr} A = \sum_{i=1}^n \lambda_i$ $\quad \operatorname{tr}(A^*A) = \sum_{i=1}^n |\lambda_i|^2$ where λ_i's are the (not necessarily distinct) eigenvalues of A.

Solution. If A is similar to B, then $\operatorname{tr} A = \operatorname{tr} B$.

(Thm 6.19 6.20* $\Rightarrow$) Diagonalize A, $P^*AP = D$.

$$\operatorname{tr} A = \operatorname{tr} D = \sum_{i=1}^n \lambda_i$$

$$\operatorname{tr}(A^*A) = \operatorname{tr}((PDP^*)^*(PDP^*)) = \operatorname{tr}(D^*D) = \sum_{i=1}^n |\lambda_i|^2$$

11. Find an orthogonal matrix whose first row is $(\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$

Solution: Extend $\{(\frac{1}{3}, \frac{2}{3}, \frac{2}{3})\}$ to a basis e.g. $\{(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}), (0,1,0), (0,0,1)\}$

Gram-Schmidt $\Rightarrow$ $\{(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}), (-\frac{2}{3\sqrt{5}}, \frac{5}{3\sqrt{5}}, -\frac{2^2}{3\sqrt{5}}), (-\frac{2}{\sqrt{5}}, 0, \frac{1}{\sqrt{5}})\}$

$$\Rightarrow \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3\sqrt{5}} & \frac{5}{3\sqrt{5}} & -\frac{2^2}{3\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \end{pmatrix}$$

15. U a unitary operator on an inner product space V. W a fin. dim. U-invariant subspace of V. Prove

a) $U(W) = W$

b) $W^\perp$ is U-invariant

Solution: a) $\| U_{|W}(x) \| = \| U(x) \| = \|x\|$ $\Rightarrow$ U_W is an unitary operator on W.

$\Downarrow$

U_W is an injection.

W is fin dim $\Rightarrow$ U_W is surjective

$U(W) = W$

b) $\forall w \in W$, $\exists y \in W$ s.t. $U(y) = w$ (a))

$x \in W^\perp$. $U(x) = w_1 + w_2$ $\quad w_1 \in W \quad w_2 \in W^\perp$. (cf. Ex 6.2.6)

U is unitary $\Rightarrow$ $\|y\|^2 = \|w\|^2$, $\quad \|x\|^2 = \|w_1 + w_2\|^2 = \|w_1\|^2 + \|w_2\|^2$ (Ex 6.1.10)

$U(x+y) = 2w_1 + w_2$

$\Rightarrow 0 = \|x+y\|^2 - \|2w_1 + w_2\|^2 = \|x\|^2 + \|y\|^2 - 4\|w_1\|^2 - \|w_2\|^2 = -2\|w_1\|^2$

$\Rightarrow w_1 = 0 \Rightarrow U(x) \in W^\perp$

24. T, U orthogonal operators on $\mathbb{R}^2$. (Thm 6.23)

a) If T and U are both reflections about lines through the origin, then UT is a rotation

b) If T is a rotation and U is a reflection about a line through the origin, then both UT and TU are reflections about lines through the origin.

Solution: a) Composition of 2 unitary operators is an unitary operator.

$\det(UT) = \det U \det T = (-1)(-1) = 1$.

$\Rightarrow$ UT is a rotation. Thm 6.23

b) Similar as above.

$\det(UT) = \det(TU) = \det T \det U = 1 \cdot (-1) = -1$

$\Rightarrow$ They are reflections

§6.6 Orthogonal projections

4. W a fin. dim subspace of an inner product space ~~of~~ V. Show that if T is the orthogonal projection of V on W, then I-T is the orthogonal projection of V on $W^\perp$.

Solution: T is an orth. proj. $\Rightarrow N(T) = R(T)^\perp \quad R(T) = N(T)^\perp$.

To prove: $N(I-T) = R(T) = W, \quad R(I-T) = N(T) = W^\perp$.

$x \in N(I-T), \Rightarrow x = T(x) \in R(T)$.

$\Rightarrow (I-T)T(x) = T(x) - T^2(x) = T(x) - T(x) = 0 \quad \Rightarrow N(I-T) = R(T) = W$

$(I-T)(x) \in R(I-T) \Rightarrow T(I-T)(x) = T(x) - T^2(x) = T(x) - T(x) = 0$

$x \in N(T) \Rightarrow T(x) = 0 \quad \Rightarrow x = (I-T)(x) \in R(I-T) \Rightarrow R(I-T) = N(T) = W^\perp$

6. T a normal operator on a fin dim inner product space. Prove that if T is a projection, then T is also an orthogonal projection.

Solution: To prove: $R(T)^\perp = N(T)$.

$x \in R(T)^\perp \qquad \langle T(x), T(x) \rangle = \langle x, T^*T(x) \rangle = \langle x, TT^*(x) \rangle = 0$

$\Rightarrow T(x) = 0$.

$x \in N(T). \qquad \langle x, T(y) \rangle = \langle T^*x, y \rangle = 0$

$T^*(x) = 0$ (Thm 6.15 c)

$\Rightarrow R(T)^\perp = N(T)$